# Stretched exponentiality and Kohlrausch-Lévy decay of optical waveguide modes

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(Received 14 May 1997)

A long-range refractive-index distribution is found to support unconventional optical modes exhibiting a stretched-exponential feature. For a certain range of parameters they show a Kohlrausch-Lévy tail along the transverse axis. It is shown that the modes include as a limiting state the algebraically decaying modes previously presented. [S1063-651X(98)06602-1]

PACS number(s): 42.65.Wi, 42.82.Et, 05.90.+m, 03.65.Ge

It was shown recently that among graded-index planar dielectric waveguides there exists a family of refractive-index distributions that make possible a bound mode at the cutoff point where the effective index coincides with the bulk index at infinity [1]. The modal fields supported by the waveguides are more weakly localized than those of above-cutoff modes in usual waveguides, in the sense that the evanescent tails of the former undergo an algebraic (a power-law) decay instead of the exponential decay of the latter. The concept of such algebraically decaying modes (ADM's) was extended to guided-wave systems allowing the multidimensional ($d$-dimensional) mode confinement [2]. The mode field profile presented therein coincides with the $t$ distribution in statistics, which includes as a special case the Cauchy distribution. Besides such an extension more comprehensive descriptions of nonextensive waveguide modes must be explored. It is expected that a guideline to seeking a generic family of such unusual modes might be gained in the context of phase transition and critical phenomena in statistical mechanics, because cutoff phenomena in guided-wave optics appear to be analogous to the statistical-mechanical phenomena. Here it should be noticed that, at a phase transition point, certain statistical-mechanical response functions universally feature a stretched-exponential decay that includes as a special case a relaxation law first observed by Kohlrausch [3,4]. Furthermore, an approach to reformulating conventional statistical mechanics in terms of generalized statistical mechanics [5] on the basis of a generalized entropy and a Lévy family of stable distributions [6] might pertain in the framework of the present physics. In this Brief Report a long-range refractive-index distribution with an algebraically decaying tail is found to support localized optical modes exhibiting the stretched exponentiality. For a certain range of parameters they show a Kohlrausch-Lévy decay along the transverse axis. It is established that the present formalism is indeed comprehensive in the sense that the Kohlrausch-Lévy-type modes include as a limiting case the ADM's [1,2] previously presented.

We consider a dielectric waveguide with a graded refractive-index profile in $d$ dimensions. With a traveling-wave phase factor

$$\exp\{i[(d-1)m\phi+n_{\mathrm{eff}}z-\omega t]\} \tag{1}$$

being implied, from Maxwell equations followed by scalar approximation [7] we obtain a Helmholtz equation for the spherically symmetric modal function $f(r)$ in $d$ dimensions:

$$f_{rr}+(d-1)r^{-1}f_r-(d-1)^2m^2r^{-2}f+[n^2(r)-n_{\mathrm{eff}}^2]f=0, \tag{2}$$

with

$$n^2(r)=n_s^2+2n_s\Delta n g(r). \tag{3}$$

Here $n_{\mathrm{eff}}$, $n_s$, and $\Delta n(\ll n_s)$ are positive constants governing, respectively, the effective refractive index, the bulk index of a substrate, and the refractive-index difference between the center and the infinity; $m$ is an arbitrary integer that indicates the number of variations along the azimuthal ($\phi$) axis of a fiber ($d=2$) [7]; $g(r)$ is a nonsingular, continuous function representing a graded refractive-index profile along the radial ($r$) axis. (Thus we exclude from our theory so-called step-index structures.) Note that for planar systems ($d=1$) the axis is reduced to a single transverse ($x$) axis, i.e., $r\equiv|x|$, where $-\infty<x<\infty$. For fiber geometries ($d=2$), $r=(x^2+y^2)^{1/2}$. In Eq. (3) we assume that $g(0)=1$ and that $g(r)\to 0$ as $r\to\infty$. Note that in Eqs. (1)–(3) the spatial coordinate is normalized by the vacuum wave number ($k_0$) as $k_0r\to r$, $k_0z\to z$.

As a generic expression of a nonsingular transverse field exhibiting a Kohlrausch-Lévy decay [3,4,6,8] as $r\to\infty$, we consider a six-parameter family

$$f(r)=f_0r^s\{\exp[-c(\alpha r^{2j}+1)^p]\}^q, \tag{4}$$

where $f_0$ is a nonvanishing constant, and $(\alpha,c,j,p,q,s)$ are positive parameters featuring details of the transverse field configuration; $j$ is a natural number (i.e., $j=1,2,\ldots$). The center value and the decaying behavior of the transverse field, respectively, are given by

$$f(0)=\begin{cases} f_0/e^{cq} & \text{for } s=0 \quad (5a)\\ 0 & \text{for } s>0, \quad (5b)\end{cases}$$

$$f(r)\sim f_0r^s\exp(-cq\alpha r^{2jp}) \quad \text{as } r\to\infty. \tag{6}$$

Note that, irrespective of the value of $s$, $df(r)/dr=0$ at the center ($r=0$). To maintain conditions of the Kohlrausch-Lévy decay [3,4,6,8] for $s=0$, we set the condition that the exponent of $r$ in the argument of the exponential function in Eq. (6) must not exceed two, i.e.,

 

$$0 < jp \leqslant 1. \tag{7}$$

Previously a two-parameter trial function similar to Eq. (6) was used in a variational calculus of modes in weakly guiding fibers with a truncated power-law refractive-index profile [9].

It is interesting to note that for $s=0$ applying a Fourier transform ($r\rightarrow\rho$) to Eq. (6) gives a Pareto distribution [8]

$$\psi(\rho)\propto\rho^{-(1+2jp)}, \tag{8}$$

where $\psi(\rho)$ represents a function in the Fourier-transformed domain.

Substituting Eq. (4) into Eq. (2) with Eq. (3), we obtain the two relations

$$(d-1)^2m^2=s(s+d-2), \tag{9}$$

$$\begin{aligned} n^2(r)-n_{\text{eff}}^2=2cjpq\alpha\{&[d+2(j+s-1)]r^{2(j-1)}C^{1-p}(r;j)\\ &+2j(p-1)\alpha r^{2(2j-1)}C^{2-p}(r;j)\\ &-2cjpq\alpha r^{2(2j-1)}C^{2(1-p)}(r;j)\}, \end{aligned} \tag{10}$$

with an extended Cauchy distribution function

$$C(r;j)=(\alpha r^{2j}+1)^{-1}\sim\alpha^{-1}r^{-2j}\quad\text{as }r\rightarrow\infty, \tag{11}$$

where $C^x(r;j)\equiv[C(r;j)]^x$. Note that $C^x(r;1)$ and $C(r;1)$, respectively, coincide with a $t$ distribution and a Cauchy distribution.

It follows from Eq. (9) that

$$s=0\ \text{ or }\ s=1\quad\text{for }d=1, \tag{12a}$$

$$s=|m|\quad\text{for }d=2. \tag{12b}$$

Note that for $d=1$, $s=0$ (1) corresponds to the lowest-order symmetric (antisymmetric) mode of a dielectric planar waveguide whose refractive-index profile is given by Eq. (10).

As $r\rightarrow\infty$, Eq. (10) becomes

$$\begin{aligned} n^2(r)-n_{\text{eff}}^2\sim 2cjpq\alpha\{&[d+2(j+s-1)]\alpha^{p-1}r^{2(jp-1)}\\ &+2j(p-1)\alpha^{p-1}r^{2(jp-1)}\\ &-2cjpq\alpha^{2p-1}r^{2(2jp-1)}\}\\ \sim -(2cjpq&\alpha^p)^2r^{2(2jp-1)}. \end{aligned} \tag{13}$$

Since it has been assumed that $g(r)\rightarrow 0$ as $r\rightarrow\infty$, from Eqs. (3) and (13) it must be required that $jp\leqslant\frac{1}{2}$. Thus, from this inequality and Eq. (7), as an allowable domain of $jp$ we derive

$$0<jp\leqslant\tfrac{1}{2}. \tag{14}$$

From Eqs. (3), (10), and (13) the effective index $n_{\text{eff}}$ and the graded-index profile $g(r)$ are determinable: For $jp=\frac{1}{2}$

$$n_{\text{eff}}=[n_s^2+(cq)^2\alpha^{1/j}]^{1/2}, \tag{15}$$

$$\begin{aligned} 2n_s\Delta ng(r)=(cq)^2\alpha^{1/j}&\\ +cq\alpha\{&[d+2(j+s-1)]r^{2(j-1)}C^{1-p}(r;j)\\ &+(1-2j)\alpha r^{2(2j-1)}C^{2-p}(r;j)\\ &-cq\alpha r^{2(2j-1)}C^{2(1-p)}(r;j)\}, \end{aligned} \tag{16}$$

otherwise (i.e., for $0<jp<\frac{1}{2}$)

$$n_{\text{eff}}=n_s, \tag{17}$$

$$\begin{aligned} 2n_s\Delta ng(r)=2cjpq\alpha\{&[d+2(j+s-1)]r^{2(j-1)}C^{1-p}(r;j)\\ &+2j(p-1)\alpha r^{2(2j-1)}C^{2-p}(r;j)\\ &-2cjpq\alpha r^{2(2j-1)}C^{2(1-p)}(r;j)\}. \end{aligned} \tag{18}$$

Therefore, from Eqs. (13), (16), and (18), as the asymptotic behavior of $g(r)$ in the limit of $r\rightarrow\infty$, we obtain

$$2n_s\Delta ng(r)$$

$$\sim\begin{cases} cq(d+2s-1)\alpha^p r^{-1} & \text{for } jp=\frac{1}{2}, \quad (19a)\\ -(2cjpq\alpha^p)^2r^{2(2jp-1)} & \text{for } 0<jp<\frac{1}{2}. \quad (19b)\end{cases}$$

It should be noted from Eq. (6) that solely for $(jp,s)=(\frac{1}{2},0)$ the transverse decay of the mode field becomes exponential [i.e., $f(r)\sim f_0\exp(-cq\alpha r)$ as $r\rightarrow\infty$].

As $r\rightarrow 0$, from Eqs. (16) and (18), respectively, it follows for $jp=\frac{1}{2}$ that

$$2n_s\Delta ng(r)\rightarrow(cq)^2\alpha^{1/j}+cq[d+2(j+s-1)]\alpha r^{2(j-1)}, \tag{20}$$

whereas for $0<jp<\frac{1}{2}$ that

$$2n_s\Delta ng(r)\rightarrow 2cjpq[d+2(j+s-1)]\alpha r^{2(j-1)}. \tag{21}$$

As a consequence, for case I: $(j,p)=(1,\frac{1}{2})$,

$$2n_s\Delta ng(0)=cq(d+cq+2s)\alpha, \tag{22}$$

for case II: $(j,p)=(2,\frac{1}{4}),\ (3,\frac{1}{6}),\ (4,\frac{1}{8}),\ldots,$

$$2n_s\Delta ng(0)=(cq)^2\alpha^{1/j}, \tag{23}$$

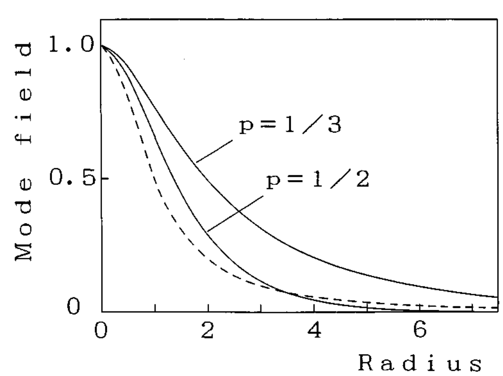


FIG. 1. Mode field distributions of Eq. (4) with $f_0=e$, $s=0$, $cq=1$, and $j=1$ vs a normalized radius, $\alpha^{1/2}r$ (solid lines). For comparison the Cauchy distribution $C(r;1)$ is plotted with a dashed line.

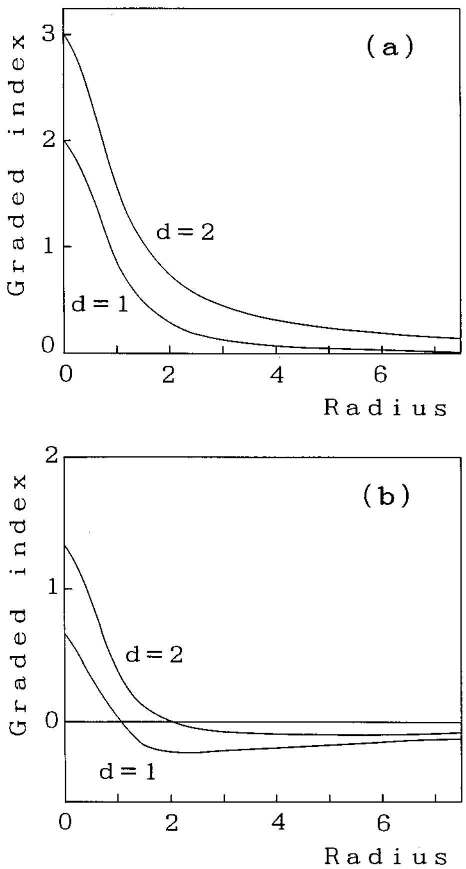


FIG. 2. Normalized refractive-index distributions of (a) Eq. (16) for $jp=\frac{1}{2}$ and of (b) Eq. (18) for $jp=\frac{1}{3}$ vs a normalized radius. Here $s=0$, $cq=1$, and $j=1$. The ordinate and the abscissa indicate $(2\alpha n_s \Delta n)^{-1}\, g(r)$ and $\alpha^{1/2} r$, respectively.

and for case III: $j=1$, $0<p<\frac{1}{2}$,

$$2n_s \Delta n g(0)=2cpq(d+2s)\alpha. \tag{24}$$

Here case IV: $0<p<1/(2j)$ for $j=2,3,4,\ldots$ is rejected because $g(0)=0$, which conflicts the assumption that $g(0)=1$. Substituting $g(0)=1$ into Eqs. (22)–(24), we obtain the relations among field $(\alpha,c,j,p,q,s)$, index $(n_s,\Delta n)$, and dimensionality $(d)$ parameters:

$$\alpha=\begin{cases} 2n_s\Delta n[cq(d+cq+2s)]^{-1} \quad \text{for case I,} & (25)\\ [2n_s\Delta n(cq)^{-2}]^{j} \quad \text{for case II,} & (26)\\ n_s\Delta n[cpq(d+2s)]^{-1} \quad \text{for case III.} & (27)\end{cases}$$

To show graphical representations of the modes, for a typical set of waveguide parameters the mode field and the refractive-index distributions are plotted with solid lines in Figs. 1 and 2, respectively.

It may be interesting to explore what happens in the limit of $p\to 0$. To this end we rewrite Eq. (4) as

$$f(r)=f_0 r^s [F(R)]^{-q}, \tag{28}$$

$$F(R)=\exp[c(\alpha R+1)^p], \tag{29}$$

where $R\equiv r^{2j}$. As $p\to 0$ while maintaining $cp=1$, Taylor expanding Eq. (29) reduces it to

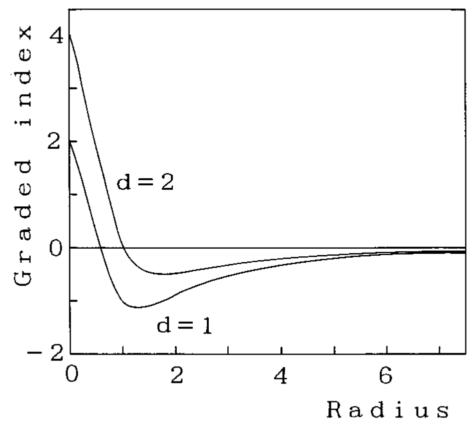


FIG. 3. Normalized refractive-index distributions of Eq. (33) with $q=1$. The ordinate and the abscissa are as in Fig. 2. The mode field profile that is supported by the index distributions is plotted with a dashed line in Fig. 1 [$f_0'=1$, $s=0$, and $q=1$ in Eq. (31)].

$$F(R)\to e^c(1+\alpha R). \tag{30}$$

Substitution of Eq. (30) into Eq. (28) yields

$$f(r)\to f_0' r^s C^q(r;j), \tag{31}$$

where $f_0'\equiv f_0 e^c$. It should be emphasized that for $(s,j)=(0,1)$ Eq. (31) coincides with the ADM's [1,2] showing a $t$ distribution that is familiar as a typical long-range distribution function in statistics. Further, for $(s,j,q)=(0,1,1)$ Eq. (31) is reduced to the Cauchy distribution.

In this limit (i.e., $p\to 0$ with $cp=1$) the refractive-index distribution of Eq. (18) can be reduced to

$$2n_s\Delta n g(r)\to 2jq\alpha\{[d+2(j+s-1)]r^{2(j-1)}C(r;j) - 2j(q+1)\alpha r^{2(2j-1)}C^2(r;j)\}. \tag{32}$$

Specifically, for $(j,s)=(1,0)$ Eq. (32) becomes

$$2n_s\Delta n g(r)\to 2q\alpha[dC(r;1)-2(q+1)\alpha r^2 C^2(r;1)], \tag{33}$$

$$2n_s\Delta n g(r)\sim 2q\alpha[d-2(q+1)]r^{-2} \quad \text{as } r\to\infty. \tag{34}$$

Note that with $q\equiv(D-2)/2$ (where $D>2$) and $\alpha\equiv a^{-2}$ Eqs. (33) and (34) reproduce the refractive-index profile of the ADM's in $d$ dimensions [2]. Also note that as in the context of fractal sciences the difference, $d-2(q+1)$, in Eq. (34) may be termed the codimension [10]. The mode field and the refractive-index distributions are shown in Figs. 1 (dashed line) and 3, respectively.

In conclusion, an algebraically decaying refractive-index distribution along the transverse (the radial) axis has been shown to support unconventional optical modes exhibiting a stretched-exponential feature. It has been found that for a certain range of shape parameters they show a Kohlrausch-Lévy decay along the transverse axis. It has also been shown that the modes presented herein include as a limiting case the ADM's previously presented.

[1] K. Hayata and M. Koshiba, Opt. Lett. **20**, 1131 (1995). [Errata: The symbol $\Delta n$ in Eqs. (2), (3), (5), (6), (8), (10), (11), (15), (16), and the caption of Fig. 1 should be replaced by $n_s \Delta n$.]

[2] K. Hayata and M. Koshiba, Opt. Rev. **2**, 331 (1995). [Errata: (1) The symbol $\Delta n$ in Eqs. (2), (3), (11), (12), and (21) should be replaced by $n_s \Delta n$. (2) The right-hand sides of Eqs. (19) and (20) should be multiplied by $f_0$, which is assumed to be real.]

[3] K. L. Ngai, Comments Solid State Phys. **9**, 127 (1979); **9**, 141 (1980).

[4] J. C. Phillips, J. Non-Cryst. Solids **172–174**, 98 (1994).

[5] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[6] M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch, *Lévy Flights and Related Topics in Physics* (Springer-Verlag, Berlin, 1995), p. VI.

[7] D. Gloge, Appl. Opt. **10**, 2252 (1971).

[8] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).

[9] K. Hayata, M. Koshiba, and M. Suzuki, Electron. Lett. **22**, 1070 (1986).

[10] H. Takayasu, *Fractals in the Physical Sciences* (Manchester University, Manchester, 1990), p. 108.